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1 Taking Derivatives

1.1 The Chain Rule

1. Let $y = \sqrt{2x+1}$. Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left[(2x+1)^{1/2}\right] = \frac{1}{2}\cdot(2x+1)^{-1/2}\cdot\frac{d}{dx}\left[2x+1\right]$$
$$= \frac{1}{\cancel{2}}(2x+1)^{-1/2}\cdot\cancel{2}$$
$$= \frac{1}{\sqrt{2x+1}}$$

2. Let $y = (1-x)^{200}$. Find $y'(x)$.

$$y'(x) = \frac{d}{dx}\left[(1-x)^{200}\right] = 200\cdot(1-x)^{199}\cdot\frac{d}{dx}(1-x)$$
$$= 200(1-x)^{199}\cdot(-1)$$
$$= -200(1-x)^{199}$$

3. Let $y = e^{1-x^2}$. Find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left[e^{1-x^2}\right] = e^{1-x^2}\cdot\frac{d}{dx}\left[1-x^2\right]$$
$$= -2x\cdot e^{1-x^2}$$

4. Let $f(x) = \ln(x^2)$. Find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[\ln(x^2)\right]$$
$$= \frac{1}{x^2}\cdot\frac{d}{dx}\left[x^2\right] = \frac{1}{x^2}\cdot 2x = \frac{2}{x}$$

5. Let $f(x) = \Big(\ln(x)\Big)^2$. Find $f'(x)$.

$$\frac{d}{dx}\left[\Big(\ln(x)\Big)^2\right] = 2\cdot\Big(\ln(x)\Big)^1\cdot\frac{d}{dx}\left[\ln(x)\right]$$
$$= 2\cdot\ln(x)\cdot\frac{1}{x}$$
$$= \frac{2\cdot\ln(x)}{x}$$

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6. Let $f(x) = \sqrt{\boxed{e^x + \sin(x) + 5x^3}}$. Find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[\left(e^x + \sin(x) + 5x^3\right)^{\frac{1}{2}} \right]$$

$$= \frac{1}{2}\left(e^x + \sin(x) - 5x^3\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left[e^x + \sin(x) - 5x^3\right]$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{e^x + \sin(x) - 5x^3}} \cdot \left(e^x + \cos(x) - 15x^2\right)$$

7. Let $f(x) = \tan(\boxed{e^x \cdot x^2})$. Find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[\tan(e^x \cdot x^2)\right]$$

chain rule

$$\left(\begin{array}{l} f(u) = \tan(u) \Rightarrow \quad f'(u) = \sec^2(u) \\ u = e^x \cdot x^2 \end{array} \right)$$

$$= \sec^2(e^x \cdot x^2) \cdot \frac{d}{dx}\left[\boxed{e^x} \cdot \boxed{x^2}\right]$$

Product rule

$$= \sec^2(e^x \cdot x^2)\left(e^x \cdot 2x + x^2 \cdot e^x\right)$$

8. Compute the derivative of $f(x) = \boxed{\cos(\boxed{x^2})}\,\boxed{\sin(\boxed{x^2})}$

product rule

Chain rule

$$f'(x) = \cos(x^2) \cdot \frac{d}{dx}\left(\sin(x^2)\right) + \sin(x^2) \cdot \frac{d}{dx}\left(\cos(x^2)\right)$$

$$= \cos(x^2) \cdot \cos(x^2) \cdot \frac{d}{dx}[x^2] + \sin(x^2)\left(-\sin(x^2)\right) \cdot \frac{d}{dx}(x^2)$$

$$= \cos^2(x^2) \cdot 2x - \sin^2(x^2) \cdot 2x$$

9. Let $f(x) = \sqrt{\boxed{\ln(x^2 + x + 1)}}$. Find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[\left(\ln(x^2 + x + 1)\right)^{\frac{1}{2}}\right]$$

$$= \frac{1}{2}\left(\ln(x^2 + x + 1)\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left(\ln(x^2 + x + 1)\right)$$

$$= \frac{1}{2} \cdot \left(\ln(x^2 + x + 1)\right)^{-1/2} \cdot \frac{1}{x^2 + x + 1} \cdot \frac{d}{dx}\left[x^2 + x + 1\right]$$

$$= \frac{2x + 1}{2\sqrt{\ln(x^2 + x + 1)} \cdot (x^2 + x + 1)}$$

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10. Let $f(x) = \sin(\cos(\tan(x)))$. Find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[\sin(\cos(\tan(x)))\right]$$

$$= \cos(\cos(\tan(x))) \cdot \frac{d}{dx}\left[\cos(\tan(x))\right]$$

$$= \cos(\cos(\tan(x))) \cdot (-\sin(\tan(x))) \cdot \frac{d}{dx}\left[\tan(x)\right]$$

$$= -\cos(\cos(\tan(x))) \cdot \sin(\tan(x)) \cdot \sec^2(x)$$

11. Let $f(x) = \cot(x \cdot \csc(2x))$. Find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[\cot(x \cdot \csc(2x))\right]$$

$$= -\csc^2(x \cdot \csc(2x)) \cdot \frac{d}{dx}\left[x \cdot \csc(2x)\right]$$

$$= -\csc^2(x \cdot \csc(2x)) \cdot \left(x\left(-\csc(2x)\cdot\cot(2x)\right)\frac{d}{dx}[2x] + \csc(2x)\cdot 1\right)$$

$$= -\csc^2(x \cdot \csc(2x)) \cdot \left(-2x \cdot \csc(2x) \cdot \cot(2x) + \csc(2x)\right)$$

12. Let $f(x) = \sec(\cos(\sin(x)))$. Find $f'(x)$.

$$f'(x) = \frac{d}{dx}\sec(\cos(\sin(x)))$$

$$= \underbrace{\sec(\cos(\sin(x))) \cdot \tan(\cos(\sin(x)))}_{\text{(outside)' at inside}} \cdot \underbrace{\frac{d}{dx}\left[\cos(\sin(x))\right]}_{\text{inside}'}$$

$$= \sec(\cos(\sin(x))) \cdot \tan(\cos(\sin(x))) \cdot \underbrace{(-\sin(\sin(x)))}_{\text{(outside)' at inside}} \cdot \underbrace{\cos(x)}_{\text{(inside)'}}$$

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1.2 Implicit Differentiation

1. Suppose that $xy = x^2$. Find $\frac{dy}{dx}$.

Short cut:

$y = x \implies \boxed{\frac{dy}{dx} = 1}$

2. Suppose that $y^2 - 4xy + x^2 = 1$. Find $\frac{dy}{dx}$.

$\frac{d}{dx}\left[y^2 - (4x)y + x^2 \right] = \frac{d}{dx}[1]$

$2y \cdot y' - \left[4x \cdot \frac{d}{dx}[y] + y \frac{d}{dx}[4x] \right] + 2x = 0$

$2y \cdot y' - 4xy' - 4y + 2x = 0$

$2y \cdot y' - 4xy' = 4y - 2x$

$y'(2y - 4x) = 4y - 2x$

$\boxed{y' = \frac{4y - 2x}{2y - 4x} = \frac{2y - x}{y - 2x}}$

3. Suppose that $e^y = \cos(x) + y$. Find $\frac{dy}{dx}$.

$\frac{d}{dx}\left[e^y\right] = \frac{d}{dx}\left[\cos(x) + y\right]$

$e^y \cdot y' = -\sin(x) + y'$

$e^y y' - y' = -\sin(x)$

$y'\left[e^y - 1\right] = -\sin(x)$

$\boxed{y' = \frac{-\sin(x)}{e^y - 1}}$

4. Suppose that $e^y = y\cos(y) + x$. Find $\frac{dy}{dx}$.

$\frac{d}{dx}\left[e^y\right] = \frac{d}{dx}\left[y \cdot \cos(y) + x\right]$

$e^y \cdot y' = y \cdot \frac{d}{dx}[\cos(\boxed{y})] + \cos(y) \cdot \frac{d}{dx}[y] + 1$

$e^y y' = y \cdot (-\sin(y)) \cdot \underbrace{\frac{d}{dx}[y]}_{y'} + \cos(y) \cdot y' + 1$

$e^y y' + y \cdot \sin(y) \cdot y' - \cos(y) \cdot y' = 1 \implies \cdots \implies y' = \frac{1}{e^y + y \cdot \sin(y) - \cos(y)}$

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Logarithms and Differentiation

1. Let $f(x) = \ln\left(\sqrt{e^x \cdot \cos(x) \cdot (x^2-1)}\right)$. Find $f'(x)$

$\ln(a^r) = r \cdot \ln(a)$

$f(x) = \ln\left((e^x \cdot \cos(x) \cdot (x^2-1))^{1/2} \right)$

$= \frac{1}{2} \cdot \ln\left(e^x \cdot \cos(x) \cdot (x^2-1)\right)$

$\ln(ab) = \ln(a) + \ln(b)$

$= \frac{1}{2} \cdot \left(\ln(e^x) + \ln(\cos(x)) + \ln(x^2-1) \right)$

$\ln(e^x) = x$

$f'(x) = \frac{d}{dx}\left[\frac{1}{2} \cdot x + \frac{1}{2} \cdot \ln(\cos(x)) + \frac{1}{2}\ln(x^2-1) \right]$

$= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{\cos(x)} \cdot (-\sin(x)) + \frac{1}{2} \cdot \frac{1}{x^2-1} \cdot 2x$

$\boxed{f'(x) = \frac{1}{2} - \frac{1}{2}\tan(x) + \frac{x}{x^2-1}}$

2. Let $f(x) = \ln\left(x^{\sin(x)}\right)$. Find $f'(x)$

$f'(x) = \frac{d}{dx}\left[\ln\left(x^{\sin(x)} \right) \right]$

$= \frac{d}{dx}\left[\sin(x) \cdot \ln(x) \right]$

$\boxed{f'(x) = \cos(x) \cdot \ln(x) + \sin(x) \cdot \frac{1}{x}}$

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3. Let $f(x) = \left(x\right)^{e^x}$. Find $f'(x)$.

$$y = (x)^{e^x}$$

$$\ln(y) = \ln\left(x^{e^x}\right) = e^x \cdot \ln(x)$$

$$\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[e^x \cdot \ln(x)\right]$$

$$\frac{1}{y} \cdot y' = e^x \cdot \frac{1}{x} + \ln(x) \cdot e^x$$

$$y' = y\left(\frac{e^x}{x} + \ln(x)\, e^x\right)$$

$$f'(x) = (x)^{e^x} \cdot \left(\frac{e^x}{x} + \ln(x)\, e^x\right)$$

4. Let $f(x) = \left(\ln(x)\right)^{\sin(x)}$. Find $f'(x)$.

$$y = \left(\ln(x)\right)^{\sin(x)}$$

$$\ln(y) = \ln\left(\left(\ln(x)\right)^{\sin(x)}\right) = \sin(x) \cdot \ln\left(\ln(x)\right)$$

product rule

$$\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[\sin(x) \cdot \ln\left(\ln(x)\right)\right]$$

$$\frac{1}{y} \cdot y' = \sin(x) \cdot \frac{d}{dx}\left[\ln\left(\ln(x)\right)\right] + \ln\left(\ln(x)\right) \cdot \cos(x)$$

chain rule

$$\cancel{y}\left(\frac{1}{\cancel{y}}\, y'\right) = \left(\sin(x) \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x} + \cos(x) \cdot \ln\left(\ln(x)\right)\right) \cdot y$$

$$\boxed{f'(x) = \left(\ln(x)\right)^{\sin(x)} \left(\frac{\sin(x)}{x \cdot \ln(x)} + \cos(x) \cdot \ln\left(\ln(x)\right)\right)}$$

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2.2 Related Rates

[See Solutions to Related Rates Worksheet]

1. An airplane flies directly over a radar station, at a constant altitude of 3 mi above the ground. A little while later, the radar station measures that (a) the distance between the plane and radar station equals 5 mi and (b) that the distance between the plane and radar station is increasing at a rate of 500 mi/hr. What is the ground speed of the airplane at the time of the second measurement?

2. An ice cube melts, with its surface area decreasing at a rate of 3 in^2/s. How fast is the side length decreasing when the side length is 1 in?

3. A streetlight is mounted at the top of a 6 meter pole, and a 2 meter tall person is walking toward it at 2 meters per second. How fast is the length of their shadow changing when they are 4 meters from the streetlight? What about when they are 1 meter from the light?

4. A police officer is walking down a city street, when they spot a wanted felon standing 200 ft away at the corner of the next block. The police officer takes off after the felon at 12 ft/s, and the felon immediately cuts around the corner and runs away at 9 ft/s. What is the rate of change of the distance between the officer and the felon after 10 seconds have passed?

5. Suppose there is a 100 cm long water trough which is empty at time $t = 0$. The cross-section of the trough is an inverted triangle ∇ which is 20 cm across the top, and is 10 cm tall. If the tank is being being filled with water at a constant rate of 400 cm^3/s, how fast is the height changing when the tank is half full?

6. Suppose the water trough above leaks (100 cm long, cross section is a ∇, top = 20 cm, and height = 1 cm). If water is being added to the tank at a rate of 400 cm^3/s, and is leaking out of the tank at 100 cm^3/s, how fast is the height changing when the tank is half full?

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Use L'Hospital's Rule to answer the following limits. Remember to **show all work**.

This is the best way to learn to do these problems correctly!

5. Does the limit $\lim_{x\to 0+} \frac{\ln(x^3)}{x^3+3}$ converge? If so, what does it converge to?

$\ln(x^3) \to -\infty$, $x^3+3 \to 3$

the limit diverges, but goes to $-\infty$. CANNOT apply L'Hopital's.

$= -\infty$

6. Does the limit $\lim_{x\to\infty} \frac{\ln(x^3)}{x^3+3}$ converge? If so, what does it converge to?

$\ln(x^3) \to \infty$, $x^3+3 \to \infty$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x^3}\cdot 3x^2}{3\cdot x^2} = \lim_{x\to\infty} \frac{1}{x^3} = 0$$

7. Does the limit $\lim_{x\to\infty} 2x \sin\left(\frac{1}{2x}\right)$ converge? If so, what does it converge to?

$2x \to \infty$, $\frac{1}{2x} \to 0$, $\sin\left(\frac{1}{2x}\right) \to 0$

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{2x}\right)}{\frac{1}{2x}} \overset{H}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{2x}\right)\cdot \cancel{\frac{-1}{2x^2}}}{\cancel{\frac{-1}{2x^2}}}$$

$$= \lim_{x\to\infty} \cos\left(\frac{1}{2x}\right) = \cos(0) = 1$$

8. Does the limit $\lim_{x\to\infty} \left(1+\frac{8}{x}\right)^x$ converge? If so, what does it converge to?

$1+\frac{8}{x} \to 1$, $x \to \infty$

indeterminate limit.

$y = \left(1+\frac{8}{x}\right)^x$.

$$\boxed{\ln(y) = \ln\left(\left(1+\frac{8}{x}\right)^x\right) = x\cdot\ln\left(1+\frac{8}{x}\right)}$$

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} x\cdot \ln\left(1+\frac{8}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{8}{x}\right)}{\frac{1}{x}} \quad \begin{matrix}\to 0\\ \to 0\end{matrix}$$

$$\lim_{x\to\infty} \ln(y) \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{8}{x}}\cdot\left(\cancel{\frac{-1}{x^2}}\right)\cdot 8}{\left(\cancel{\frac{-1}{x^2}}\right)} = \frac{1\cdot 8}{1} = 8$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln(y)} = \boxed{e^8}$$

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4. Compute the limit $\lim_{x\to\infty}\left(1+\frac{8}{x}\right)^x$ $\Rightarrow$ type 1^∞

$y = \left(1+\frac{8}{x}\right)^x$

$\ln(y) = \ln\left(\left(1+\frac{8}{x}\right)^x\right) = x\cdot\ln\left(1+\frac{8}{x}\right)$

$\lim_{x\to\infty}\ln(y) = \lim_{x\to\infty} x\cdot\ln\left(1+\frac{8}{x}\right)$

$y=\ln(x)$

$= \lim_{x\to\infty}\frac{\ln\left(1+\frac{8}{x}\right)}{\frac{1}{x}}$

$\lim_{x\to\infty}\ln(y) \overset{H}{=} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{8}{x}}\cdot\frac{-8}{\cancel{x^2}}}{\frac{-1}{\cancel{x^2}}} = 8$

$= \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln(y)} = \boxed{e^8}$

Answer: e^8

5. Compute the limit $\lim_{x\to 0^-}(1+3\sin(x))^{2/x}$ type 1^∞

$\ln(y) = \ln\left[(1+3\sin(x))^{2/x}\right]$

$\lim_{x\to 0^-}\ln(y) = \lim_{x\to 0^-}\frac{2}{x}\cdot\ln(1+3\sin(x))$

$= \lim_{x\to 0^-} 2\cdot\frac{\ln(1+3\sin(x))}{x}$

$\overset{H}{=} \lim_{x\to 0^-} 2\cdot\frac{\frac{1}{1+3\sin(x)}\cdot 3\cos(x)}{1}$

$\lim_{x\to 0^-}\ln(y) = 2\cdot 3 = 6$

$= \lim_{x\to 0^-} y = \lim_{x\to 0^-} e^{\ln(y)} = \boxed{e^6}$

answer: e^6

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3 Abstract Applications of Derivatives

3.1 Linear Approximations

1. Approximate $e^{-0.1}$ using the method of linear approximation. You must give your answer as a decimal without using a calculator.

① find function:

$f(x) = e^x$

② find nearby point (we know e^0)

$a=0$

③ find tangent line at nearby pt

$L(x) = f'(0)\cdot(x-0) + f(0)$

$= 1(x-0) + 1$

$L(x) = x+1$

$f'(x) = e^x$

$f'(0) = e^0 = 1$

$f(0) = e^0 = 1$

④ plug in -0.1

$e^{-0.1} \approx L(-0.1)$

$= -0.1 + 1 = 0.9$

$\boxed{e^{-0.1} \approx 0.9}$

2. Approximate $\sqrt{8}$ using the method of linear approximation. You must give your answer as a decimal without using a calculator.

① find function

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2}\cdot(x)^{-\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}}$

$f'(9) = \frac{1}{2}\cdot\frac{1}{\sqrt{9}} = \frac{1}{6}$

② find nearby point (we know $\sqrt{9}$)

$a = 9$

$f(9) = \sqrt{9} = 3$

③ find tangent line at nearby pt

$L(x) = f'(9)(x-9) + f(9)$

$L(x) = \frac{1}{6}(x-9) + 3$

④ plug in 8

$\sqrt{8} \approx L(8) = \frac{1}{6}(8-9)+3 = \frac{-1}{6}+3 = 3-1.66\ldots = 2.833\ldots$

$\boxed{\sqrt{8} \approx 2.833}$

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3. Find the linear approximation to $f(x) = \frac{1}{x} + x$ at $a = 2$. Use your answer to approximate $f(1.9)$. You must give your answer as a decimal without using a calculator.

$L(x)$ = tangent to $f(x)$ at 2

$= f'(2)(x-2) + f(2)$

$f'(x) = \frac{-1}{x^2} + 1$

$f'(2) = \frac{-1}{4} + 1 = 0.75$

$f(2) = \frac{1}{2} + 2 = 2.5$

$L(x) = 0.75(x-2) + 2.5$

because 1.9 is near 2

$f(1.9) \approx L(1.9)$

$= 0.75(1.9-2) + 2.5$

$= 0.75(-0.1) + 2.5$

$= 2.5 - 0.075$

$= 2.425$

$\boxed{f(1.9) \approx 2.425}$

4. Find the linear approximation to $f(x) = x\ln(x^2)$ at $a = 1$. Use your answer to approximate $f(1.2)$. You must give your answer as a decimal without using a calculator.

$L(x)$ = tangent to $f(x)$ at 1

$= f'(1)(x-1) + f(1)$

product Rule

$f'(x) = \ln(x^2) + x \cdot \frac{1}{x^2} \cdot 2x$

$= \ln(x^2) + 2$

$f'(1) = \ln(1^2) + 2 = 2$

$f(1) = 1 \cdot \ln(1^2) = 1 \cdot 0 = 0$

$L(x) = 2(x-1) + 0$

$= 2x - 2$

Because 1.2 is near 1

$f(1.2) \approx L(1.2)$

$= 2(1.2) - 2$

$= 2.4 - 2$

$= 0.4$

$\boxed{f(1.2) \approx 0.4}$

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Abstract Applications of Derivatives

$$L(x) = f'(a)(x-a) + f(a)$$

1. Find a linear approximation for the function $f(x) = \sin(x)$ at $a = \frac{\pi}{4}$.

   Use your answer to approximate $\sin\left(\frac{5\pi}{16}\right)$.

$f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = \frac{d}{dx}(\sin(x)) = \cos(x)$

$f'\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\Rightarrow L(x) = \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}$

Because $\frac{5\pi}{16} \approx \frac{4\pi}{16} = \frac{\pi}{4}$

$f\left(\frac{5\pi}{16}\right) \approx L\left(\frac{5\pi}{16}\right) = \frac{\sqrt{2}}{2}\left(\frac{5\pi}{16} - \frac{4\pi}{16}\right) + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \cdot \frac{\pi}{16} + \frac{\sqrt{2}}{2}$

$\boxed{\sin\left(\frac{5\pi}{16}\right) = f\left(\frac{5\pi}{16}\right) \approx \frac{\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{32}}$

2. Find a linear approximation for the function $f(x) = \cos(x)$ at $a = \frac{\pi}{4}$

   Use your answer to approximate $\cos\left(\frac{5\pi}{16}\right)$.

$f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = -\sin(x)$

$f'\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = \frac{-\sqrt{2}}{2}$

$\Rightarrow L(x) = \frac{-\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}$

Because $\frac{5\pi}{16} \approx \frac{4\pi}{16} = \frac{\pi}{4}$

$f\left(\frac{5\pi}{16}\right) \approx L\left(\frac{5\pi}{16}\right) = \frac{-\sqrt{2}}{2}\left(\frac{5\pi}{16} - \frac{4\pi}{16}\right) + \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2} \cdot \frac{\pi}{16} + \frac{\sqrt{2}}{2}$

$\boxed{\cos\left(\frac{5\pi}{16}\right) = f\left(\frac{5\pi}{16}\right) \approx \frac{\sqrt{2}}{2} - \frac{\pi\sqrt{2}}{32}}$

3. Find a linear approximation for the function $f(x) = \tan(x)$ at $a = \frac{\pi}{4}$

   Use your answer to approximate $\tan\left(\frac{5\pi}{16}\right)$.

$f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

$f'(x) = \frac{d}{dx}\tan(x) = \sec^2(x)$

$f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = \frac{1}{\left(\cos\left(\frac{\pi}{4}\right)\right)^2} = \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} = \frac{1 \cdot 4}{\frac{2}{4} \cdot 4} = \frac{4}{2} = 2$

$\Rightarrow L(x) = 2 \cdot \left(x - \frac{\pi}{4}\right) + 1$

Because $\frac{5\pi}{16} \approx \frac{4\pi}{16} = \frac{\pi}{4}$, $f\left(\frac{5\pi}{16}\right) \approx L\left(\frac{5\pi}{16}\right) = 2\left(\frac{5\pi}{16} - \frac{4\pi}{16}\right) + 1$

$= 2 \cdot \frac{\pi}{16} + 1 = 1$

$\Rightarrow \boxed{\tan\left(\frac{5\pi}{16}\right) \approx 1 + \frac{\pi}{8}}$

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4. Let $f(x) = x^3 - 6x^2 + 9x + 1$

   Find the following. If a requested quantity doesn't exist, answer "DNE".

   (a) The intervals where $f(x)$ is increasing/decreasing. Identify which is which.

   (b) The intervals where $f(x)$ is concave up/down. Identify which is which.

   (c) The x value(s) of the local maxima and local minima of f. Identify which is which.

   (d) The x value(s) of the inflection points of f.

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$$

$$f'(x) = 3(x-1)(x-3)$$

(a) f is increasing $\Leftrightarrow$ $f'(x)$ is positive

$(-)(-)$ $(+)$	1	$(+)(-)$ $(-)$	3	$(+)(+)$ $(+)$

f is increasing on $(-\infty, 1) \cup (3, \infty)$

f is decreasing on $(1, 3)$

$$f''(x) = 6x - 12 = 6(x-2)$$

(b) f is concave up $\Leftrightarrow$ f'' is positive

$(-)$	2	$(+)$

f is concave up on $(2, \infty)$

f is concave down on $(-\infty, 2)$

(c) local max (↗ ↘) at 1

local min (↘↗) at 3

local max/min when change increasing/decreasing

(d) inflection point $\Leftrightarrow$ change concavity

inflection point at $x = 2$

Name: ______________________ Section: ______________________

2. Let $f(x) = x^4 - 8x^2 + 16$.

Find the following. If a requested quantity doesn't exist, answer "DNE".

(a) Find the intervals where f is increasing and decreasing

$$f'(x) = 4x^3 - 16x = 4x(x^2-4) = 4x(x+2)(x-2)$$

$f'(x) = 0$ when $x = 0, 2, -2$

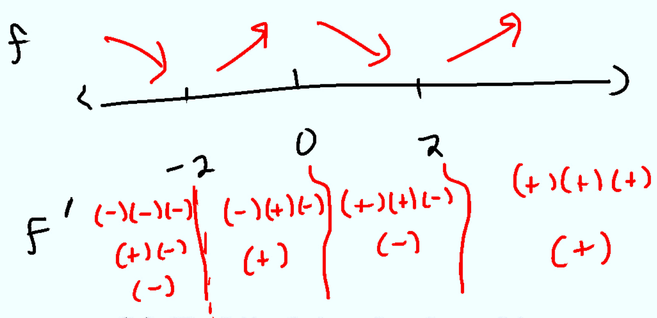
$f'(x)$ always exists

increasing on

$(-2, 0) \cup (2, \infty)$

decreasing on

$(-\infty, -2) \cup (0, 2)$



(b) Find the intervals where f is concave up and concave down

$$f''(x) = 12x^2 - 16$$

$f''(x) = 0$ when

$16 = 12x^2$

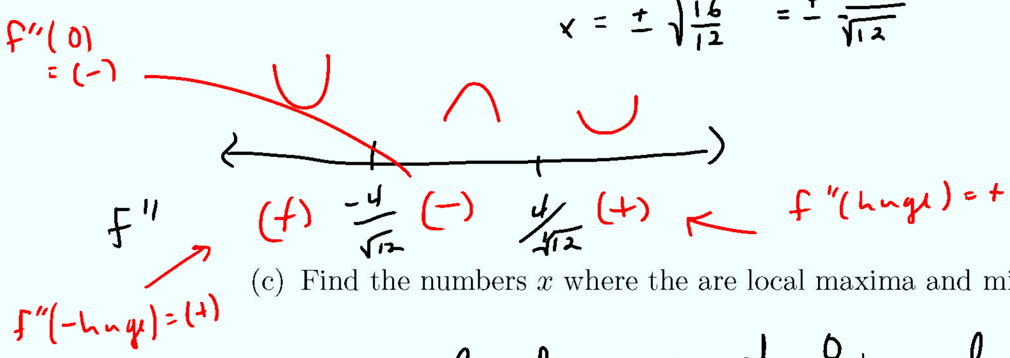
$x = \pm\sqrt{\frac{16}{12}} = \pm\frac{4}{\sqrt{12}}$

Concave up on

$\left(-\infty, \frac{-4}{\sqrt{12}}\right) \cup \left(\frac{4}{\sqrt{12}}, \infty\right)$

concave down on

$\left(\frac{-4}{\sqrt{12}}, \frac{4}{\sqrt{12}}\right)$



(c) Find the numbers x where the are local maxima and minima

local max at 0, local min at -2 & 2

(d) Find the numbers x where the is a point of inflection

inflection pt at $\pm\sqrt{\frac{16}{12}}$

Name: ______________________ Section: ______________________

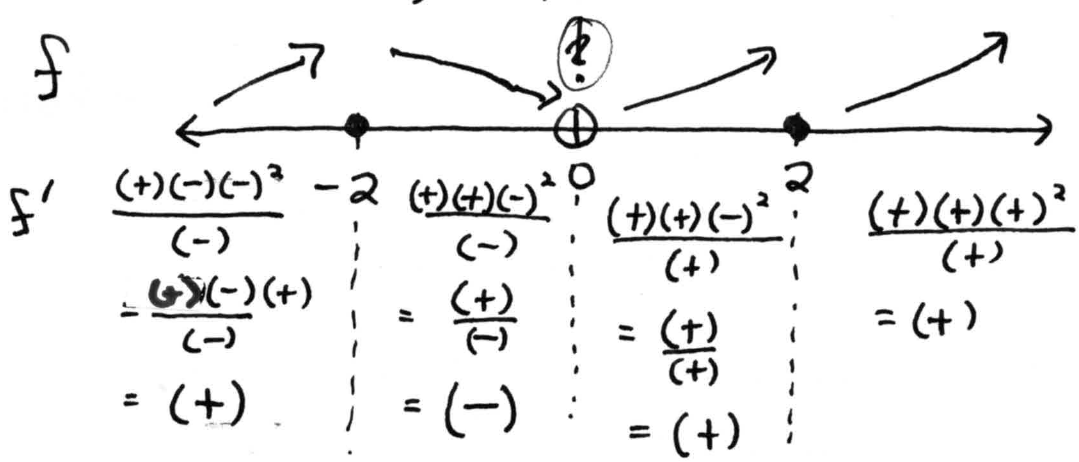
3. Let f be a mystery function with domain $(-\infty, 0) \cup (0, \infty)$. Suppose also that

$$f'(x) = \frac{e^x (x+2)(x-2)^2}{x}$$

(a) Find the intervals where f is increasing and decreasing.

$f'(x) = 0$ when $x = -2$ or $x = 2$

$f'(x)$ DNE when $x = 0$



f increasing on $(-\infty, -2) \cup (0, 2) \cup (2, \infty)$

f decreasing on $(-2, 0)$

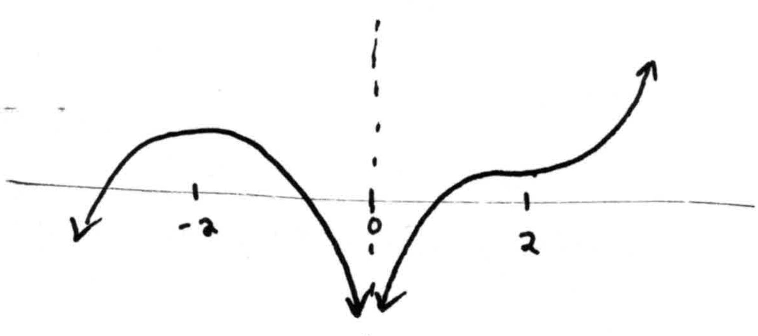
(b) Find the numbers x where f has local maxima and minima.

BOTH f' changes sign **AND** f exists

local max at $x = -2$

NO local min ← [$f(0)$ DNE and f' doesn't change sign at 2]

(c) Sketch a graph of f.



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2.3 Optimization

1. In optimization problems, it is important to verify that you have the correct answer.

   (a) Find two positive numbers whose sum is 25 and whose product is as large as possible.

Maximize product subject to the constraint

$F = x \cdot y$

$x + y = 25$

$\Rightarrow y = 25 - x$

$F = x(25 - x)$

$F(x) = 25x - x^2 \Leftarrow$ maximize this

Check to verify max.

$F'(x) = 25 - 2x$

critical #'s.
F' always defined
$F'(x) = 0$ when $x = \frac{25}{2}$

F: ↗ ↘

F': $(+)$ $\frac{25}{2}$ $(-)$ ✓

product is maximized when $x = \frac{25}{2}$ and $y = 25 - \frac{25}{2} = \frac{25}{2}$

   (b) Find two *positive* numbers whose sum is 25 and whose product is as *small* as possible.

Minimize product subject to the constraint

$x + y = 25$

NOTICE from above: NO absolute min for product when x ranges over $(-\infty, \infty)$.

Rereading question $\Rightarrow$ NOTICE additional constraint

x & y positive $\Rightarrow$ $x \geq 0$ & $y \geq 0$

so $x \geq 0$ and $x \leq 25$

$\Rightarrow$ use test for absolute max/min on $[0, 25]$

$\Rightarrow$ test f of endpoints & critical #'s.

$f(0) = 0 \cdot (25 - 0) = 0$

$f(\frac{25}{2}) = \frac{25}{2}(25 - \frac{25}{2}) = \frac{(25)^2}{4}$

$f(25) = 0(25 - 25) = 0$

on $[0,25]$, Product is minimized when $x = 0$ and $y = 25$

Name: ____________________ Section: ____________________

4. Find two numbers x and y such that $xy^2 = 54$ and which minimizes $F = x^2 + y^2$.

Minimize Subject to Constraint

$F = x^2 + y^2$

$xy^2 = 54$

$\Rightarrow y^2 = \frac{54}{x}$

$F(x) = x^2 + \frac{54}{x}$

$\Uparrow$ minimize this

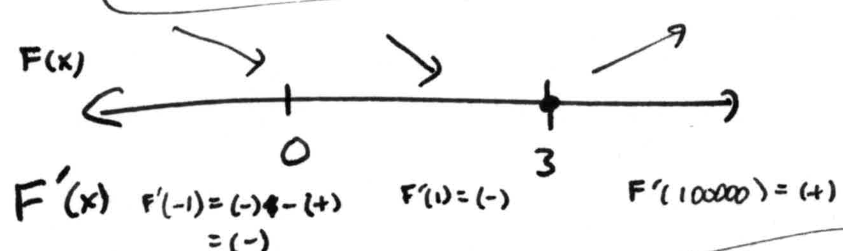
$F'(x) = 2x - \frac{54}{x^2}$

$F'(x)$ DNE when $x = 0$

$F'(x) = 0$ when $2x = \frac{54}{x^2}$

$x^3 = 27$

$x = 3$



$F(x)$ is minimized when $x = 3$

and $y = \sqrt{\frac{54}{3}}$

$= \sqrt{18} = \sqrt{2 \cdot 9} = 3\sqrt{2}$

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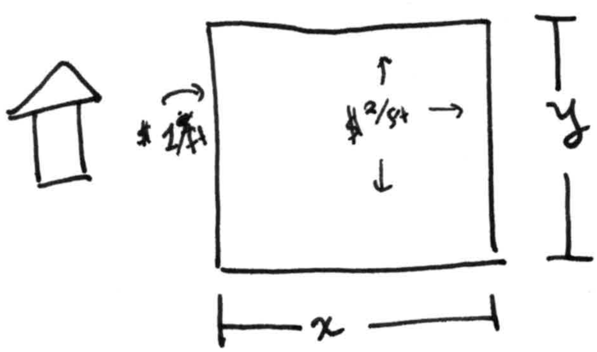
5. You are building a rectangular garden. You are tired of your neighbors stealing your produce and you have looked into several types of electric fence. The material for the side of the fence near your house is \$1 per foot. The other three sides will cost \$2 per foot. If you have \$100, what is the maximum amount of area you can protect?

   You must **show all work**, including verifying that area is maximized.



$\text{Cost} = \$1(y) + \$2(x + x + y)$

$= y + 4x + 2y$

$= 3y + 4x$

Maximize area subject to condition: cost = \$100

$A(x) = xy$

$100 = 3y + 4x$

$3y = 100 - 4x$

$y = \frac{100}{3} - \frac{4}{3}x$

$A(x) = x\left(\frac{100}{3} - \frac{4}{3}x\right)$

$A(x) = \frac{100}{3}x - \frac{4}{3}x^2 \Leftarrow$ maximize this.

$A'(x) = \frac{100}{3} - \frac{8}{3}x$

Critical #'s

A' always def'd

$A'(x) = 0$ when $\frac{100}{3} = \frac{8}{3}x$

$\frac{100}{8} = \frac{25}{2} = x$

Check to verify max

A: increasing, then decreasing

A': (+) $\frac{25}{2}$ (−)

$A'(0) = (+)$ $A'(100) = (-)$

$\boxed{\text{max area when } x = \frac{25}{2} \text{ and } y = 50 - 2\cdot\left(\frac{25}{2}\right),\ y = 25}$

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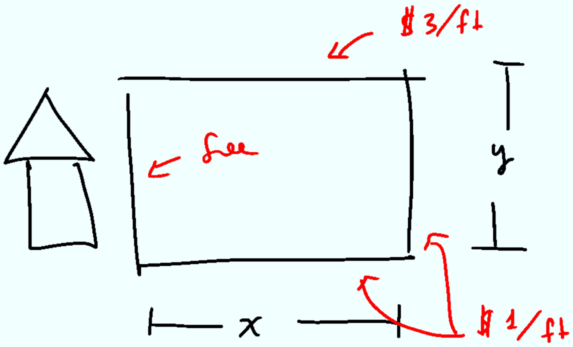
4. You want to turn part of your yard into a dog park. For zoning reasons, you want the park to be exactly 100 ft^2. Suppose that your house will form the west boundary of the yard, that the fence for the north boundary will cost \$3 per foot and that the other two sides will cost \$1 per foot. What are the dimensions of the yard with the cheapest surrounding fence?

   You must **show all work**, including verifying that cost is minimized.



minimize cost subject to CONSTRAINT

$C = 0 + 3\cdot x + 1\cdot y + 1\cdot x$

$C = 4x + y$

$C(x) = 4x + \frac{100}{x}$

Area $= xy = 100$

$y = \frac{100}{x}$

$C'(x) = 4 + \frac{-100}{x^2}$

$C'(x)$ DNE when $x=0$

$C'(x)=0$ when

$4 = \frac{100}{x^2}$

$4x^2 = 100$

$x^2 = 25$

$x = \pm 5$

C	↗	↘	↘	↗
	-5	0	5	
C'	(+)	(-)	(-)	(+)

$\underbrace{(-)\quad(+)}_{\text{physical solutions}}$

cost is minimized when

$x = 5$

$y = \frac{100}{5} = 20$

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6. You are designing a rectangular display box with an open top. The box should have a volume of 2 m^3, and the length of the base should equal twice the width of the base. The material for the base costs \$5 per square meter, and the material for the sides costs \$10 per square meter. Find the cost of the materials for the cheapest container.

You must **show all work**, including verifying that this cost is a minimum.

minimize cost subject to constraints

① volume = 2

$h \cdot l \cdot w = 2$

AND

② $\boxed{l = 2w}$ useful

so

$h \cdot (2w) \cdot w = 2$

so $\boxed{h = \frac{1}{w^2}}$

$$\text{Cost} = 10\underbrace{(hl + hw + hl + hw)}_{\text{area of sides}} + 5\underbrace{(lw)}_{\text{area of base}}$$

$$\text{Cost} = 10(2hl + 2hw) + 5lw$$

$$C(w) = 20\cdot\left(\frac{1}{w^2}\right)(2w) + 20\left(\frac{1}{w^2}\right)w + 5(2w)w$$

$$C(w) = \frac{40w}{w^2} + \frac{20w}{w^2} + 10w^2$$

$$C(w) = \frac{60}{w} + 10w^2 \Leftarrow \text{minimize this}$$

$$C'(w) = \frac{-60}{w^2} + 20w$$

critical #'s:

C' DNE when $w = 0$

$C' = 0$ when $\frac{60}{w^2} = 20w$

$60 = 20w^3$

$3 = w^3$

$w = \sqrt[3]{3}$

verify min

C'	(-)	0	(-)	$\sqrt[3]{3}$	(+)

Cost is minimized when

$w = \sqrt[3]{3}$

$\Rightarrow$ min cost is

$$C(\sqrt[3]{3}) = \frac{60}{\sqrt[3]{3}} + 10(\sqrt[3]{3})^2$$

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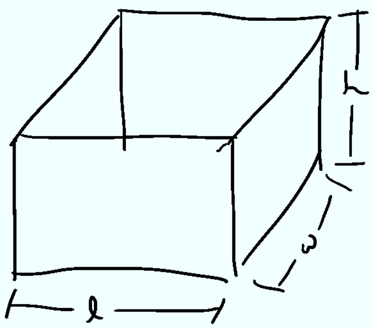
6. You are designing a rectangular display box with an open top. The width of the base must equal twice its length. The material for the base costs \$6/in^2 and the material for the sides costs \$2/in^2. What is the largest volume box that you can make for \$120?

   You must **show all work**, including verifying that volume is maximized.



maximize volume subject to CONSTRAINT

$w = 2\ell$

Cost $= 120$

$= 6 \cdot \ell w + 2(2\ell h + 2wh)$

$= 6 \cdot \ell(2\ell) + 4\ell h + \underbrace{4(2\ell)h}_{8\ell h}$

$120 = 12\ell^2 + 12\ell h$

$10 = \ell^2 + \ell h$

$h = \dfrac{10-\ell^2}{\ell}$

$V = \ell \cdot w \cdot h$

$V = \cancel{\ell}(2\ell) \cdot \dfrac{10-\ell^2}{\cancel{\ell}}$

$V(\ell) = 20\ell - 2\ell^3$

$V'(\ell) = 20 - 6\ell^2$

$$\boxed{\begin{array}{l} V'(\ell) = 0 \text{ when } 20 = 6\ell^2 \\ \quad \text{when } \ell = \pm\sqrt{\frac{20}{6}} \\ V'(\ell) \text{ always exists} \end{array}}$$

V

V' (−) $-\sqrt{\frac{20}{6}}$ (+) 0 $\sqrt{\frac{20}{6}}$ (−)

physically possible

max vol at $\ell = \sqrt{\frac{20}{6}}$

max vol

$V\left(\sqrt{\frac{20}{6}}\right) = 20\left(\sqrt{\frac{20}{6}}\right) - 2\left(\sqrt{\frac{20}{6}}\right)^3$

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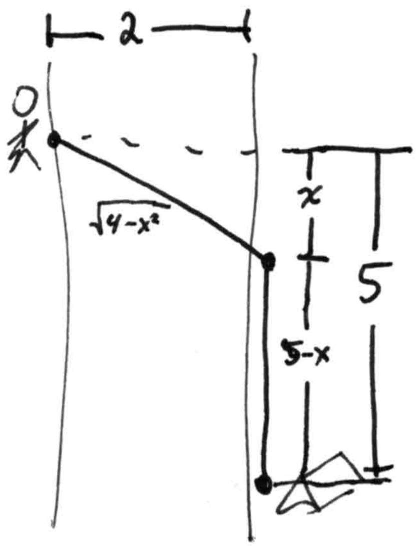
3. You are crossing a 2 mile wide river. Your camp is on the other side, 5 miles downstream. You are hungry, and you want to get there as quickly as possible. If you can swim at 3 mi/hr and hike at 5 mi/hr, where should you land to get there the fastest?

You must **show all work**, including verifying that time is minimized.

minimize time subject to constraint ~~get there~~

limited speed.



distance is $\sqrt{4+x^2} + 5 - x$

speed $= \dfrac{\text{distance}}{\text{time}} \Rightarrow$ time $= \dfrac{\text{distance}}{\text{speed}}$

time is

$$f(x) = \frac{\sqrt{4+x^2}}{3} + \frac{5-x}{5}$$

$$= \frac{1}{3}(4+x^2)^{\frac{1}{2}} + 1 - \frac{1}{5}x$$

minimize $f(x)$

$$f'(x) = \frac{1}{3}\cdot\frac{1}{\cancel{2}}\cdot(4+x^2)^{-\frac{1}{2}}\cdot\cancel{2}x - \frac{1}{5}$$

$$= \frac{1}{3}\cdot\frac{x}{\sqrt{4+x^2}} - \frac{1}{5}$$

$f'(x)$ always def'd.

$f'(x) = 0$ when

$$\frac{1}{5} = \frac{1}{3}\frac{x}{\sqrt{4+x^2}}$$

$$3\sqrt{4+x^2} = 5x$$

$$9(4+x^2) = 25x^2$$

$$36 + 9x^2 = 25x^2$$

$$36 = 16x^2$$

$$x^2 = \frac{36}{16}$$

$$x = \frac{6}{4} = \frac{3}{2}$$

land 1.5 miles downstream

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3.4 Anti-Derivatives

1. Find the (general) anti-derivative of $f(x) = \sin(x) + \pi\cos(x) + \frac{3\pi}{2}\sec^2(x)$.

$\Rightarrow \quad F(x) = -\cos(x) + \pi\cdot\sin(x) + \frac{3\pi}{2}\cdot(\tan(x)) + C$

2. Find the (general) anti-derivative of $f(x) = \dfrac{x^2+x+1}{x^3} + \dfrac{1}{\sqrt{x}}$.

REWRITE: $f(x) = \frac{x^2}{x^3} + \frac{x}{x^3} + \frac{1}{x^3} + \frac{1}{\sqrt{x}} = \frac{1}{x} + x^{-2} + x^{-3} + x^{-\frac{1}{2}}$

$\Rightarrow \quad F(x) = \ln|x| + \frac{x^{-2+1}}{-2+1} + \frac{x^{-3+1}}{-3+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \boxed{\ln|x| - \frac{1}{x} - \frac{1}{2x^2} + 2\sqrt{x} + C}$

3. Suppose $f'(x) = x^2 + 2x + 5$ and that $f(0) = 3$. Find $f(x)$.

$\Rightarrow f(x) = \frac{x^3}{3} + x^2 + 5x + C$

$f(0) = 3 = \frac{0^3}{3} + 0^2 + 5\cdot 0 + C$

$3 = C$

$f(x) = \frac{x^3}{3} + x^2 + 5x + 3$

4. Suppose $f'(x) = \frac{1}{x} + x + e^x$ and that $f(1) = 0$. Find $f(x)$.

$\Rightarrow f(x) = \ln|x| + \frac{x^2}{2} + e^x + C \quad = \ln|x| + \frac{x^2}{2} + e^x - \frac{1}{2} - e$

$f(1) = 0 = \ln|1| + \frac{(1)^2}{2} + e^1 + C$

$0 = 0 + \frac{1}{2} + e + C$

$C = -\frac{1}{2} - e$

5. Suppose $f''(x) = 12x^2 - e^x$, that $f(0) = 1$, and that $f'(0) = 2$. Find $f(x)$.

$f'(x) = \frac{12}{3}x^3 - e^x + C = 4x^3 - e^x + C$

$f'(0) = 2 = 4\cdot 0^3 - e^0 + C$

$2 = 0 - 1 + C$

$C = 3$

$f'(x) = 4x^3 - e^x + 3$

$\Rightarrow f(x) = \frac{4x^4}{4} - e^x + 3x + D$

$= x^4 - e^x + 3x + D$

$f(0) = 1 = 0^4 - e^0 + 3\cdot 0 + D$

$1 = -1 + D$

$D = 2$

$\boxed{f(x) = x^4 - e^x + 3x + 2}$